Minimal irreversible quantum mechanics. The decay of unstable states.

Diego G. Arbó,[1,2,3] Mario A. Castagnino,[1,2,4] Fabián H. Gaioli,[1,5] Sergio Iguri[1]

[1]Instituto de Astronomía y Física del Espacio, C. C. 67, Suc. 28, 1428 Buenos Aires, Argentina

[2]Consejo Nacional de Investigaciones Científicas y Técnicas, Av. Rivadavia 1917, 1033 Buenos Aires, Argentina

[3]Departamento de Ingeniería e Investigaciones Tecnológicas, Universidad Nacional de La Matanza, F. Varela 1903, 1754 S. Justo, Buenos Aires, Argentina

[4]Departamento de Física, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, 1428 Buenos Aires, Argentina

[5]Departamento de Física, Universidad Nacional del Sur, Alem 1253, 8000 Bahía Blanca, Buenos Aires, Argentina

Abstract

Brownian motion is modelled by a harmonic oscillator (Brownian particle) interacting with a continuous set of uncoupled harmonic oscillators. The interaction is linear in the coordinates and the momenta. The model has an analytical solution that is used to study the time evolution of the reduced density operator. It is derived in a closed form, in the one-particle sector of the model. The irreversible behavior of the Brownian particle is described by a reduced density matrix.

I. INTRODUCTION

In previous works [1–4] we have studied the time evolution of a quantum oscillator coupled to a dense, but discrete (finite) bath of harmonic oscillators. For such system an irreversible behavior has appeared as a consequence of averaging in time the evolution of the characteristic quantum-oscillator variables (macroscopic quantities), since time evolution splits in very different scales: One related to small fluctuations, which are erased by averaging, another one related with recurrence phenomena, which are far enough of laboratory observational times, and the last one connected with observable phenomena, which involves irreversibility.

In Ref. [1] it has been shown how to pass to the continuous bath. A resonance, which can be isolated and leads to an evolution for a macroscopic period of time (the same period as in the discrete case), has arisen because of the energy of the quantum oscillator is embedded in the continuum. One particularity of this continuous limit is that two of the three time scales have become irrelevant since they have zero measure with respect to the remaining time scale. However, this last scale is not exclusively governed by the resonance (associated with an exponential decay) but also by contributions coming from the semibounded feature of the energy spectrum.

Leaving these contributions aside the main behavior of the quantum oscillator is an exponential decay towards the equilibrium with the bath, and can be described only with the contribution due to the resonance. The present work is the conclusion of the previous papers [5–8], where a novel method was used to work directly in the continuum, including the exponential decay law in quantum mechanics. In this paper we continue with the development of the formalism we have called "Minimal irreversible quantum mechanics," where time asymmetry can be represented through the choice of a subspace of "admissible" or "regular" solutions of the evolution equation.

The main idea goes as follows. Usually rigorous quantum mechanics must be formulated in a Gel'fand triplet [40]

$$\mathcal{S} \subset \mathcal{H} \subset \mathcal{S}^{\times}, \tag{1}$$

where:

$\bullet\mathcal{S}$ is the space of "regular states" or test-functions space, corresponding to Schwarz-class wave functions, that are considered as the "physical" states.

$\bullet\mathcal{H}$ is the space of "states," or Hilbert space, introduced to extend the notion of probability to a larger space and to use the well-known spectral theory of Hilbert spaces. These states correspond to square-integrable wave functions.

$\bullet\mathcal{S}^{\times}$ is the space of "generalized states", or rigged Hilbert space, namely the space of linear (or antilinear) functionals over $\mathcal{S}$, which are essentially used to find the spectral expansion of the regular states (e.g. Fourier expansions).

Let K be the Wigner or time-inversion operator. As usual the evolution Hamiltonian H is time symmetric, i.e.

$$KHK^{\dagger} = H. \tag{2}$$

In the wave function representation the action of K coincides with the complex conjugation, so it is defined over $\mathcal{S}$ by

$$K\varphi(x) = \varphi^{*}(x). \tag{3}$$

Thus

$$K : \mathcal{S} \rightarrow \mathcal{S}. \tag{4}$$

Therefore $\mathcal{S}$ is also time symmetric.

But the real universe and macroscopic objects have clearly time-asymmetric evolutions. Therefore the task of this paper and the preceding ones is to explain how this time asymmetry appears while the quantum mechanical laws of the universe (embodied in $\mathcal{H}$) are time symmetric. The usual and successful explanation is based on coarse-graining: Macroscopic

objects have a huge number of dynamical variables and we can measure and control only a small number of them, the so-called relevant variables. If we neglect the rest of the variables, the irrelevant ones, we obtain time-asymmetric evolution equations. Nevertheless in paper [5] (according to the line of thought pioneered in Refs. [9], [10], [11]) we have follow a different way. We have developed a sort of minimal irreversible quantum description, which reproduces time asymmetry from the basic microscopic level directly, where the key point is the presence of resonances (and additional hypotheses we have extracted from Ref. [3]).

Obviously we want to obtain the standard results making minimal changes to the well established and usual quantum mechanics. If we change Eqs. (2) or (3), it is almost sure to find experimental problems. So the minimal modification is to change Eq. (4) defining a new test-functions space $\Phi_+ \subset \mathcal{S}$ such that

$$K : \Phi_+ \rightarrow \Phi_- \neq \Phi_+. \tag{5}$$

In this way K is not even defined over the space of regular states Φ_+ and a time-asymmetric evolution arises.

This can be done if we postulate, as we have done in Ref. [5],[1] that all the "regular" or "admissible" states belong to a space $\mathcal{H}_+ \sim \theta(H^2_+)$ and also to $\mathcal{S}$. Then $\Phi_+ \sim \theta(H^2_+ \cap \mathcal{S})$ [the time inverted states belong to a space $\mathcal{H}_- \sim \theta(H^2_-)$ and $\Phi_- \sim \theta(H^2_- \cap \mathcal{S})$, respectively],where θ is the Heaviside step function that gives the restriction to the positive real energy axis and $H^2_\pm$ are the Hardy class function spaces [9].[2]

An "irreversible" quantum theory based on a Gel'fand triplet

$$\Phi_\pm \subset \mathcal{H}_\pm \subset \Phi^\times_\pm, \tag{6}$$

is feasible and it yields physical results, as the dominant experimental decay of unstable states, if the test-function space Φ_+ is so chosen. This will be valid for systems where the

[1]This postulate has been motivated in cosmological-global considerations in Refs. [12] and [13].

[2]As spaces $\mathcal{H}_-$ and $\mathcal{H}_+$ are isomorphic they are normally called $\mathcal{H}$ [14].

existence of resonances dominates the evolution for the relevant period of observational time. We have shown that, what it is done in the quoted papers [9], [10], and [11] is essentially a minimal modification of the ordinary reversible quantum theory. In fact, from now on we will consider that:

•Φ_+ is the space of "regular states" or test-functions space, that are considered as the "admissible" states.

•$\mathcal{H}_+$ is the space of "states," or Hilbert space. These states are again particular square-integrable wave functions. But in paper [5] and in this work we consider that only Φ_+ contains the "admissible" states.

•$\Phi_+^\times$ is the space of "generalized states," or rigged Hilbert space, namely the space of linear (or antilinear) functionals over Φ_+, which are essentially used to find the spectral expansion of the "regular states" in Sec. III.

The spaces with subscripts "–" contain the time-inverted states of the corresponding spaces with subscripts "+".

Friedrichs model [28] was studied using this approach. In this work, we show that this idea can be used to take a slightly different point of view in studying dissipation phenomena of quantum Brownian motion. This more complex model will force us to generalize the definition of space Φ_+ although the roles played by the characters in the triplets $\Phi_\pm \subset \mathcal{H}_\pm \subset \Phi_\pm^\times$ will remain the same.

Brownian motion has been extensively studied in the literature (we will only quote those papers particularly relevant to our line of work). E.g., in Ref. [15] it was shown that for a system composed by a finite number of linear interacting oscillators a dissipative behavior can be found in the limit of a dense system (continuous spectrum). But, in this work we are concerned directly with dense systems with continuous spectrum. The presence of this continuous spectrum allows us to study the decay processes using analytical properties familiar in scattering theory [6,16]. The model is a very well-known and widely used system, consisting of a harmonic oscillator coupled to an infinite and continuous bath. In this paper, as in Refs. [15,17], the bath is composed by an infinite collection of harmonic oscillators

and the interaction is modelled to be linear and characterized by the spectral weight, but otherwise arbitrary. We show that the oscillator reaches a final equilibrium state via a damped evolution which is mostly exponential. We also show that some deviations from this exponential decay law (for very short and very long times) appear, which are intimately related with the presence of a lower bound of the energy.

In Sec. II the whole system (single oscillator plus the bath) is described and the Hamiltonian is introduced.

In Sec. III we diagonalize (in normal modes) the Hamiltonian. In the process of diagonalization some problems emerge, such as the lost of the discrete part of the energy spectrum [28,18]. We can bypass these problems, if we use our definition of "regular" states. Then we can perform an analytical continuation of the spectral decomposition of the Hamiltonian, promoting the energy to complex values. To reach a successful interpretation of the results we require to generalize the definition given in paper [5] to the model we are now studying. The mathematical bases of this generalization are shown in Appendix B.[3] This appendix also contains a rigorous mathematical understanding of the problem.

In Sec. IV mixed states and their evolution law are considered.

In Sec. V we deal with a very particular initial condition: An oscillator in a zero-temperature bath. We find the reduced density operator and show that the equilibrium state is reached. We accurately describe the time evolution of the system and estimate the Zeno [19] and Khalfin [20] effects for very short and long times, respectively. These are the deviations from an exact exponential decay law. Finally we show that our solution satisfy a Lindblad master equation when discarding these deviations from the exponential behavior.

Finally in Sec. VI some questions concerning irreversibility, already considered in papers [21] and [5], are discussed.

---

[3]The reader who is not familiar with rigged Hilbert spaces and functional analysis can see Refs. [9] and [40].

We state our conclusions in Sec. VII.

Three mathematical appendices complete this work.

II. PARTICLE-BATH MODEL

The system is a Brownian particle represented by a harmonic oscillator with natural frequency Ω. It is well known that for a finite bath it is not possible to prove convergence in an equilibrium state in the limit $t \to \infty$ because of the existence of recurrences [15,24–26]. However, for large systems these recurrence times become extremely huge and we can eliminate them by passing to the limit of an infinite continuous bath. Therefore, in this paper, we consider the oscillator in contact with a bath, already modeled by a continuous set of harmonic oscillators with natural frequencies ω. The coupling between the system and the bath is assumed to be linear with strength $g(\omega)$. The Hamiltonian for the composite system, in terms of creation and annihilation operators, is

$$H = \Omega a^{\dagger} a + \int_0^{\infty} d\omega \omega b_{\omega}^{\dagger} b_{\omega} + \lambda \int_0^{\infty} d\omega g(\omega) \left(a^{\dagger} b_{\omega} + b_{\omega}^{\dagger} a \right) . \tag{7}$$

The first term corresponds to the system, the second to the bath, and the third one corresponds to the interaction between them. In order that the Hamiltonian would be positive definite we must require [21,27] that $g(0) = 0$ and

$$\Omega > \lambda^2 \int_0^{\infty} d\omega \frac{g^2(\omega)}{\omega}. \tag{8}$$

This is an important condition which selects the kind of spectral densities appropriated to lead to an irreversible evolution. For example, the ohmic case which is frequently used in the literature must be disregarded, unless a cutoff is used. (Operators b_{ω} and $b_{\omega}^{\dagger}$ are rigorously defined in Appendix A).

The Fock basis is the tensor product of the Fock basis of the isolated harmonic oscillator and those of the bath, namely

$$|n,\omega_1...\omega_m\rangle = |n\rangle \otimes |\omega_1...\omega_m\rangle\ , \tag{9}$$

where $|\omega_1...\omega_m\rangle$ represents a state with m quanta in the bath, each one with frequency ω_j $(j = 1, ..., m)$.

The total number of quanta is conserved allowing us to solve the problem by sectors (block diagonalization). The one-particle sector is referred as Friedrichs' model [28] and contains the relevant information that we need to compute physical quantities (see Sec. V).

III. NORMAL MODES OF THE HAMILTONIAN AND ANALYTIC CONTINUATION

The linearity in the coupling term of H allows us to easily find a new set of uncoupled harmonic oscillators (normal modes), such that

$$I = \int_0^\infty d\omega\ \widetilde{b}_\omega^\dagger \widetilde{b}_\omega, \tag{10}$$

$$H = \int_0^\infty d\omega\ \omega \widetilde{b}_\omega^\dagger \widetilde{b}_\omega, \tag{11}$$

where

$$\widetilde{b}_\omega = \xi_\omega a + \int_0^\infty d\omega' \Phi_\omega(\omega') b_{\omega'}. \tag{12}$$

From a straightforward calculation [6,10,21], using the Heisenberg equations of motion, we obtain the coefficients of the unitary change of variables which diagonalize the Hamiltonian, precisely

$$\Phi_\omega(\omega') = \delta(\omega - \omega') + \frac{\lambda \xi_\omega g(\omega')}{(\omega - \omega' + i\varepsilon)} \tag{13}$$

and

$$\xi_\omega = \frac{\lambda g(\omega)}{\alpha(\omega + i\varepsilon)}, \tag{14}$$

where

$$\alpha(z) = z - \Omega - \lambda^2 \int_0^\infty d\omega \frac{g^2(\omega)}{z - \omega}. \tag{15}$$

This function, which is the inverse of the reduced resolvent of H in the one-particle sector, is not entire because it has a cut along the positive real axis corresponding to the continuous spectrum of the Hamiltonian. If $\alpha(z) \neq 0$ for all $z \in \mathbf{C}$, except for a possible real and negative ω_0 such that $\alpha(\omega_0) = 0$, an isolated solution appears, which is non-analytic in λ . We do not consider this case henceforth, since we are interested in analytic solutions satisfying condition (8).

If $\alpha(z) = 0$ has no real solution it is not possible to find an operator $\tilde{a}$, such that $\tilde{a} \to a$ for $\lambda \to 0$. In this case we have lost the particle number operator corresponding to the discrete part of the spectrum of H and we do not have the correct form of H when $\lambda \to 0$ [30]. This problem can be solved promoting the energy (or frequency) ω to be a complex variable z. We define $\beta(z) \equiv [\alpha(z)]^{-1}$. It can be proved that $\beta(z)$ has the same analytic structure than the one of the coefficient $S(z)$ of the scattering matrix [9]. $\beta(z)$ is a meromorphic function on a double Riemann sheet with a cut along $[0, +\infty)$. $\beta_\pm(\omega) = [\alpha_\pm(\omega)]^{-1} \equiv \beta(\omega \pm i\varepsilon)$ are defined on the upper and lower half-planes of the first Riemann sheet R_I (physical sheet), and have meromorphic continuations to the lower and upper half-planes, respectively, in the second sheet R_{II} (unphysical sheet). For simplicity we consider $g(z)$ such that the analytic extension of $\beta_+(z)$ into the second sheet has a simple pole $z_0 = \omega_0 - \frac{i}{2}\gamma$ [$\gamma > 0$ and $\alpha_+(z_0) = 0$] in the lower half-plane. Also $\beta_-(z)$ has a simple pole z_0^* on the upper plane in R_{II}.

We can now study the meaning of z_0. From the role played by z_0 in the evolution equation we know that $(\mathrm{Im} z_0)^{-1} = \gamma^{-1}$ is the mean life time of the unstable state $|1, v\rangle = a^\dagger |0, v\rangle$ and $(\mathrm{Re} z_0)$ is the shift of the bare frequency Ω [see [5] and also Eq. (40)]. But, z_0 is the root of $\alpha_+(z)$ and from Eq. (15) we can estimate, up to the second order in λ,

$$z_0 = \Omega + \lambda^2 \mathrm{P} \int_0^\infty d\omega \frac{g^2(\omega)}{\Omega - \omega} - i\pi\lambda^2 g^2(\Omega), \tag{16}$$

where P denotes the Cauchy principal part of the integral.[4] The mean life of the unstable state and the shift frequency are given by

$$\gamma = 2\pi\lambda^2 g^2(\Omega) \tag{17}$$

and

$$\delta\Omega = \lambda^2 \mathrm{P} \int_0^\infty d\omega \frac{g^2(\omega)}{\Omega - \omega}. \tag{18}$$

which are well-known expressions in the theory of unstable systems [31], usually derived from the Fermi golden rule.

Regarding the coupling function of the form $g(\omega) \sim \omega^n$ we find that the ohmic case $(n = 1)$ without cutoff does not satisfy the positivity condition (8). If we call $\gamma_{1/2}$ the coefficient for the subohmic case $(0 < n < 1)$ and γ_2 the coefficient for supraohmic case $(n > 1)$, it is easy to prove that

$$\gamma_2 \ll \gamma_1 \ll \gamma_{1/2}. \tag{19}$$

Now we will find a generalized partition of the identity I and a generalized spectral decomposition of H that recovers the discrete part of the spectrum [10,21,29]. In order to do this let Γ be the curve of Fig. 1. It lays on R_I for $\beta_-(z)$ and on R_{II} for $\beta_+(z)$. We define the analytic function of $z \in \mathbf{C}$

$$\alpha_\Gamma(z) = z - \Omega - \lambda^2 \int_\Gamma dz' \frac{g^2(z')}{z - z'}, \tag{20}$$

---

[4]The principal part comes from the well known identity between distributions

$$\frac{1}{x + i\varepsilon} = \mathrm{P}\frac{1}{x} - i\pi\delta(x),\ x \in \mathbf{R}$$

which generalizes Eq. (15).

To find the partition of the identity and a expansion of H we will use in some adequate analyticity properties.[5] Thus we will define a space $\Phi_+ \subset \mathcal{H}_+$ of states $|\varphi\rangle$ such that the function $\langle 0,\omega_1,...,\omega_n|\varphi\rangle = \varphi_0(\omega_1,...,\omega_n)$ would have an analytic continuation, for each variable ω_i $(1 \leq i \leq n)$ to a region that include the singularity z_0. This space Φ_+ would be our space of "regular," "admissible" or "physical" states. Precisely, generalizing what we have done in paper [5], we will chose Φ_+ such that its states would satisfy Eq. (B3) of Appendix B. Analogously, the space Φ_- of the "unphysical" time-inverted states would satisfy Eq. (B4) of that appendix. With this choice the analytic continuation that we will perform has a rigorous meaning, since the operators act in a space Φ_+which endowed with adequate analytic properties. The demonstration of this fact is a mathematical problem, which is considered in Appendices B and C, where we generalize previous results. Then if $|\varphi\rangle \in \Phi_+$ and $|\psi\rangle \in \Phi_-$, from Eqs. (10) and (11) using appendix B and following the similar demonstration of paper [5], it can be proved that

$$\langle\psi|\varphi\rangle = \langle\psi|(\widetilde{a}^{(-)\star}\widetilde{a}^{(-)} + \int_\Gamma dz\ \widetilde{b}_z^{(-)\star}\widetilde{b}_z^{(-)})|\varphi\rangle, \tag{21}$$

$$\langle\psi|H|\varphi\rangle = \langle\psi|(z_0\widetilde{a}^{(-)\star}\widetilde{a}^{(-)} + \int_\Gamma dz\ z\widetilde{b}_z^{(-)\star}\widetilde{b}_z^{(-)})|\varphi\rangle. \tag{22}$$

The residue at z_0 contributes to the first terms of the r.h.s. of these generalized partition of the identity and spectral decomposition of H, as in paper [10], and, in a weak sense, the two previous equations can be written as

$$I = \widetilde{a}^{(-)\star}\widetilde{a}^{(-)} + \int_\Gamma dz\ \widetilde{b}_z^{(-)\star}\widetilde{b}_z^{(-)} \tag{23}$$

$$H = z_0\widetilde{a}^{(-)\star}\widetilde{a}^{(-)} + \int_\Gamma dz\ z\widetilde{b}_z^{(-)\star}\widetilde{b}_z^{(-)}, \tag{24}$$

---

[5]Properties of this kind were already introduced in previous works [5,9,11,21].

The creation and annihilation operators in all these equations reads

$$\widetilde{a}^{(-)} = \frac{1}{\sqrt{\alpha'_+(z_0)}}\left[a + \lambda\int_0^\infty dz \frac{g(z)}{[z_0 - z]_+} b_z\right],$$
$$\widetilde{a}^{(-)\star} = \frac{1}{\sqrt{\alpha'_+(z_0)}}\left[a^\dagger + \lambda\int_0^\infty dz \frac{g(z)}{[z_0 - z]_+} b_z^\dagger\right], \tag{25}$$

and

$$\widetilde{b}_\omega^{(-)} = b_\omega + \frac{\lambda g(\omega)}{\eta_+(\omega)}\left[a + \lambda\int_0^\infty d\omega' \frac{g(\omega')}{\omega - \omega' + i\varepsilon} b_{\omega'}\right],$$
$$\widetilde{b}_\omega^{(-)\star} = b_\omega^\dagger + \frac{\lambda g(\omega)}{\alpha_-(\omega)}\left[a^\dagger + \lambda\int_0^\infty d\omega' \frac{g(\omega')}{\omega - \omega' - i\varepsilon} b_{\omega'}^\dagger\right]. \tag{26}$$

The distribution $\frac{1}{[z_0 - z]_+}$ means

$$\int_0^\infty \frac{f(\omega)}{[z_0 - \omega]_+} d\omega = \int_\Gamma \frac{f(z)}{z_0 - z} dz = \int_0^\infty \frac{f(\omega)}{z_0 - \omega} d\omega + 2\pi i f(z_0), \tag{27}$$

for every well-behaved analytical function $f(z)$. Observe that $\widetilde{b}_\omega^{(-)\star}$ does not change if we replace $\int_\Gamma$ by $\int_0^\infty$ because, in this case, no pole is crossed, since $\alpha_-(z)$ has not poles (in S_I). Nevertheless $\widetilde{b}_\omega^{(-)}$ does change because the pole is crossed when we modify the integration contour. We have shown this fact explicitly putting $\eta_+(\omega)$ in place of $\alpha_+(\omega)$, where

$$\frac{1}{\eta_+(\omega)} = \frac{1}{\alpha_+(\omega)} + 2\pi i \frac{\delta(z - z_0)}{\alpha'_+(z_0)}, \tag{28}$$

being $\delta(z - z_0)$ the extension of the Dirac delta defined *à la* Gel'fand and Shilov [32,33]. As a consequence of these facts $\tilde{a}^{(-)\star} \neq \tilde{a}^{(-)\dagger}$ and $\widetilde{b}_\omega^{(-)\star} \neq \widetilde{b}_\omega^{(-)\dagger}$. The star operation corresponds to the analytic generalization of the complex conjugation, which, acting on an analytic function $f(z)$, is defined by[6]

$$f^\star(z) = [f(z^*)]^*. \tag{29}$$

[6] It corresponds to the symbol # of paper [5]. Here we follow the notation $\star$ of paper [21] in which we have also studied this model.

From Eqs. (25) and (26) we see that we have four annihilation operators due to the presence of complex eigenvalues of H with the corresponding doubling of solutions, since we have a pair of complex conjugate values. They are generalized eigenvalues of the two analytic continuations of H into the lower $(-)$ [upper $(+)$] -complex plane. These operators are

$$\widetilde{a}^{(-)};\ \widetilde{a}^{(+)};\ \widetilde{b}_{\omega}^{(-)};\ \widetilde{b}_{\omega}^{(+)}$$

The vacuum is the state annihilated by any annihilation operator. The Bogolubov transformation of Eqs. (25) and (26) does not mix creation and annihilation operators, therefore the vacuum just defined is actually the same state defined as the vacuum of the noninteracting system+bath. So from Eq. (9) the vacuum is the state $|0\rangle \otimes |v\rangle \equiv |0, v\rangle$, where $|v\rangle$ is the vacuum of the bath.

The corresponding creation operators are

$$\widetilde{a}^{(-)\star};\ \widetilde{a}^{(+)\star};\ \widetilde{b}_{\omega}^{(-)\star};\ \widetilde{b}_{\omega}^{(+)\star}$$

Starting from the common vacuum, by applying successively the operators $\widetilde{a}^{(-)\star}$ and $\widetilde{b}_{\omega}^{(-)\star}$, the Fock basis $\{|\widetilde{\ }\rangle\}$ is built, and with $\widetilde{a}^{(+)\star}$ and $\widetilde{b}_{\omega}^{(+)\star}$ we build up the Fock basis $\{|\ \ \rangle\}$. In the case of $\widetilde{a}^{(-)\star}$ and $\widetilde{a}^{(+)\star}$ the corresponding vectors in the Fock bases of the one-particle sector are generalized eigenvectors of H with purely complex eigenvalues. They represent unstable states, i.e. $\widetilde{a}^{(-)\star}|\widetilde{0, v}\rangle = \left|\widetilde{1, v}\right\rangle$ is a one-particle generalized eigenvector of H corresponding to a complex eigenvalue z_0 and $\widetilde{a}^{(+)\star}\left|\overline{0, v}\right\rangle = \left|\overline{1, v}\right\rangle$ is a one-particle generalized eigenvector of H corresponding to a complex eigenvalue z_0^*. In this way we are able to develop a second quantized version of the theory of unstable states [21].

Now we have two different number of quanta operators, $\widetilde{N}^{(-)} = \int_0^\infty d\omega \widetilde{b}_{\omega}^{(-)\star}\widetilde{b}_{\omega}^{(-)}$ and $\widetilde{N}^{(+)} = \int_0^\infty d\omega \widetilde{b}_{\omega}^{(+)\star}\widetilde{b}_{\omega}^{(+)}$, which are not Hermitian. So two different Fock bases can be built satisfying

$$\widetilde{N}^{(-)}\left|\widetilde{n, \omega_1 ... \omega_m}\right\rangle = m \left|\widetilde{n, \omega_1 ... \omega_m}\right\rangle, \tag{30}$$

$$\widetilde{N}^{(+)}\,|\overline{n,\omega_1...\omega_m}\rangle\ = m\,|\overline{n,\omega_1...\omega_m}\rangle\,.$$

The spectral decomposition of the Hamiltonian reads

$$H^{(-)} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\int_0^{\infty}...\int_0^{\infty} d\omega_1...d\omega_m\,(z_0 n + \omega_1 + ... + \omega_m) \\ \times |n,\widetilde{\omega_1...\omega_m}\rangle\,\langle\overline{n,\omega_1...\omega_m}|\,, \tag{31}$$

which acts on the right of the Fock space generated by basis $\{|\tilde{}\rangle\}$. But the "same" Hamiltonian can also be written in the following way (using the other analytical continuation, in which case it is evident that the next equation is only weak, and it has a precise meaning operating between $|\varphi\rangle \in \Phi_-$ and $|\psi\rangle \in \Phi_+$)

$$H^{(+)} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\int_0^{\infty}...\int_0^{\infty} d\omega_1...d\omega_m\,(z_0^* n + \omega_1 + ... + \omega_m) \\ \times |\overline{n,\omega_1...\omega_m}\rangle\,\langle n,\widetilde{\omega_1...\omega_m}|\,, \tag{32}$$

which acts on the right of the Fock space generated by basis $\{|\ \rangle\}$. In the same way, the identity reads

$$I^{(-)} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\int_0^{\infty}...\int_0^{\infty} d\omega_1...d\omega_m\,|n,\widetilde{\omega_1...\omega_m}\rangle\,\langle\overline{n,\omega_1...\omega_m}|\,,$$

$$I^{(+)} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\int_0^{\infty}...\int_0^{\infty} d\omega_1...d\omega_m\,|\overline{n,\omega_1...\omega_m}\rangle\,\langle n,\widetilde{\omega_1...\omega_m}|\,. \tag{33}$$

As the eigenvalues of Eqs. (31) and (32) are complex, in order to deal with unstable states we must find an adequate mathematical structure beyond the Hilbert space. In fact, these states are generalized states. Thus, in the Appendix B we see that kets $|\tilde{}\rangle$ and $|\ \rangle$ are well defined in a rigged Hilbert space formalism, i.e. they must be thought as antilinear functionals acting on test spaces $\Phi_\pm$ and, as elements of a vector space, they belong to the duals of $\Phi_\pm$, symbolized by $\Phi_\pm^\times$. They define a double Gel'fand triplet structure $\Phi_\pm \subset \mathcal{H}_\pm \subset \Phi_\pm^\times$ [9,14,21].

What about energy conservation? The trouble emerges because the eigenvalues of the Hamiltonian are now complex; thus some states decay in time (e.g. vectors of $\Phi_+^\times$ which

vanish for long times, see Sec. V). Since some states vanish we may ask ourselves how the conservation of energy can be possible. The answer is that energy is conserved anyhow. In order to demonstrate this fact we will calculate the mean value of the Hamiltonian in a state $|\varphi(t)\rangle = \sum\limits_{nm} \int\limits_0^\infty ... \int\limits_0^\infty d\omega_1...d\omega_m c_n(\omega_1...\omega_m) \, |n, \widetilde{\omega_1...\omega_m}\rangle$, precisely

$$\begin{aligned}
E &= \left\langle \varphi(t)|H^{(-)}|\varphi(t)\right\rangle \\
&= \sum_{nn'} \sum_{mm'} \int\limits_0^\infty ... \int\limits_0^\infty d\omega_1...d\omega_m d\omega_1'...d\omega_{m'}' c_n^*(\omega_1...\omega_m) c_{n'}(\omega_1'...\omega_{m'}') \\
&\quad \times e^{-i(z_0 n' + \omega_1' + ... + \omega_{m'}')t} e^{i(z_0^* n + \omega_1 + ... + \omega_m)t} (z_0 n + \omega_1 + ... + \omega_m) \\
&\quad \times \left\langle n, \widetilde{\omega_1, ...\omega_m} | n', \widetilde{\omega_1'...\omega_{m'}'} \right\rangle .
\end{aligned} \tag{34}$$

Taking into account the orthogonality relations (B7) and (B9) shown in Appendix B, and that $z_0 = \omega_0 - i\frac{\gamma}{2}$, $\gamma > 0$, Eq. (34) reduces to

$$E = \sum_m \int\limits_0^\infty ... \int\limits_0^\infty d\omega_1...d\omega_m \left| c_0(\omega_1...\omega_m) \right|^2 (\omega_1 + ... + \omega_m), \tag{35}$$

which is time independent. Thus energy is conserved. Conservation of the norm and the number of particles can also be demonstrated in an analogous way changing H by I.

Finally we can observe that Eqs. (B7) and (B9) show that the generalized eigenvectors have null norm and energy (with the exception of those with $n = 0$) [5,7,21,34]. In the literature they are called Gamow vectors, they are generalized states, and they represent just idealized mathematical states (see Appendix B), as is the case of the plane waves.

IV. MIXED STATES: ITS EVOLUTION

A general pure state belonging to Φ_+ (see Appendix B) can be written as

$$|\Psi\rangle = \sum_n \sum_m \int_0^\infty ... \int_0^\infty d\omega_1...d\omega_m \; c_n(\omega_1, ..., \omega_m) \, |n, \widetilde{\omega_1...\omega_m}\rangle, \tag{36}$$

where $|n,\widetilde{\omega_1...\omega_n}\rangle \in \Phi_+^\times$, and the most general density[7] matrix can be written as

$$\rho = \sum_{nn'}\sum_{mm'}\int_0^\infty ... \int_0^\infty d\omega_1...d\omega_m d\omega_1'...d\omega_{m'}' c_{nn'}\left(\omega_1,...,\omega_m,\omega_1',...,\omega_{m'}'\right)$$
$$\times |n,\widetilde{\omega_1...\omega_m}\rangle\langle n,\widetilde{\omega_1',...,\omega_{m'}'}|. \tag{37}$$

If this ρ is the initial state $\rho = \rho(0)$, the evolution law of $\rho(t)$ reads

$$\rho(t) = e^{-iH^{(-)}t}\rho(0)e^{iH^{(+)}t}. \tag{38}$$

As H is only self-adjoint in a generalized way [8] [9] $H^{(-)}$ acts in a different way than $H^{(+)} = H^{(-)\dagger}$ and there are right and left eigenvalues,

$$H^{(-)}|n,\widetilde{\omega_1...\omega_m}\rangle = (z_0 n + \omega_1 + ... + \omega_m)|n,\widetilde{\omega_1...\omega_m}\rangle,$$
$$\langle n,\widetilde{\omega_1...\omega_m}|\, H^{(+)} = (z_0^* n + \omega_1 + ... + \omega_m)\,\langle n,\widetilde{\omega_1...\omega_m}|\,. \tag{39}$$

Then we have

$$\rho(t) = \sum_{nn'} e^{-\frac{\gamma}{2}(n+n')t}e^{-i\omega_0(n-n')t}\sum_{mm'}\int_0^\infty ... \int_0^\infty d\omega_1...d\omega_m d\omega_1'...d\omega_{m'}'$$
$$\times e^{-i(\omega_1+...+\omega_m)t}e^{i(\omega_1'+...+\omega_{m'}')t}c_{nn'}(\omega_1,...,\omega_m,\omega_1',...,\omega_{m'}')$$
$$\times|n,\widetilde{\omega_1...\omega_m}\rangle\langle n',\widetilde{\omega_1'...\omega_{m'}'}|. \tag{40}$$

For an arbitrary initial state $\rho(0)$ a time dependent asymptotic ($t \to +\infty$) state is reached. The explanation of this fact is simple. The modes of the bath are independent of each other [see Eq. (7)], and so we cannot expect that the bath reaches equilibrium (cf. Ref. [6]). Thus

[7]More accurately, we would say, that it is the most general possible decaying density matrix, as we will see.

[8]Recall that H is self-adjoint in the Hilbert space, where $\mathcal{H} = \mathcal{H}^\times$, but in the generalized Hilbert space this property essentially becomes Eq. (39).

$$\rho(t) \to \rho_*(t) = \sum_{mm'} \int_0^\infty ... \int_0^\infty d\omega_1 ... d\omega_m d\omega'_1 ... d\omega'_{m'} e^{-i(\omega_1 + ... + \omega_m)t} e^{i(\omega'_1 + ... + \omega'_{m'})t}$$
$$\times c_{00}(\omega_1, ..., \omega_m, \omega'_1, ..., \omega'_{m'}) |0, \widetilde{\omega_1 ... \omega_m}\rangle \left\langle 0, \widetilde{\omega'_1 ... \omega'_{m'}} \right| . \tag{41}$$

For completeness we also write the evolution equation for the density operator,

$$\frac{d\rho(t)}{dt} = -i \sum_{nn'} e^{-\frac{\gamma}{2}(n+n')t} e^{-i\omega_0(n-n')t} \sum_{mm'} \int_0^\infty ... \int_0^\infty d\omega_1 ... d\omega_m d\omega'_1 ... d\omega'_{m'}$$
$$\times e^{-i(\omega_1 + ... + \omega_m)t} e^{i(\omega'_1 + ... + \omega'_{m'})t} (z_0 n + \omega_1 + ... + \omega_m - z_0^* n' - \omega'_1 - ... - \omega'_{m'})$$
$$\times c_{nn'}(\omega_1, ..., \omega_m, \omega_1, ..., \omega_{m'}) |n, \widetilde{\omega_1 ... \omega_m}\rangle \left\langle n', \widetilde{\omega'_1 ... \omega'_{m'}} \right| , \tag{42}$$

which is clearly equal to

$$\frac{d\rho(t)}{dt} = -i \left(H^{(-)} \rho - \rho H^{(+)} \right) = -iL\rho, \tag{43}$$

where L is the generalized Liouvillian operator [21]. So we see that the density operator follows an evolution described by a generalized Liouville-von Neumann equation.

In spite of the result obtained in Eq. (41) in Sec. V we show that the reduced density operator ρ_r, which is obtained by taking the partial trace with respect to the environment modes, reaches equilibrium, namely a time independent state. An equivalent way to find the equilibrium state, closer to the spirit of our formalism, is to use a particular space of observables, as in paper [35].

V. REDUCED DENSITY OPERATOR

As an illustration of the formalism we consider a simple example where the initial state is a very particular state of the composed system,

$$\rho(0) = \rho_S(0) \otimes \rho_E(0), \tag{44}$$

where

$$\rho_S(0) = c_{11} \left|1\right\rangle \left\langle 1\right| + c_{10} \left|1\right\rangle \left\langle 0\right| + c_{01} \left|0\right\rangle \left\langle 1\right| + c_{00} \left|0\right\rangle \left\langle 0\right| \tag{45}$$

is the initial state of the discrete oscillator, or the initial reduced density operator, (with $c_{11}, c_{00} \geq 0$, $c_{11} + c_{00} = 1$ and $c_{10} = c_{01}^*$) and

$$\rho_E(0) = \left|v\right\rangle \left\langle v\right| \tag{46}$$

is the initial state of the bath, which does not have any quantum, namely, it is in the ground state. This corresponds to a bath at zero temperature $T = 0$ (in the zero and one-particle sector). The main features at any T can be reproduced but we begin with this example because the mathematical computations are easier (recall that this model can be decomposed in sectors of constant number of quanta). Also our initial conditions are such that there is no correlation between the oscillator and the bath.

Our aim is to find the time dependence of the reduced matrix elements. It is derived from the time evolution of the density operator

$$\rho(t) = e^{-iH^{(-)}t}\rho(0)e^{iH^{(+)}t} = e^{-iH^{(-)}t}I^{(-)}\rho(0)I^{(+)}e^{iH^{(+)}t}, \tag{47}$$

where $I^{(-)}$ and $I^{(+)}$ are the identities in spaces Φ_+ and Φ_- respectively[9] [see Eqs. (33)].

$$\begin{aligned}\rho(t) = &\left(e^{-iz_0 t} \left|\widetilde{1,v}\right\rangle \left\langle \overline{1,v}\right| + \int_0^\infty d\omega \; e^{-i\omega t} \left|\widetilde{0,\omega}\right\rangle \left\langle \overline{0,\omega}\right|\right) \rho(0) \\ &\times \left(e^{iz_0^* t} \left|\overline{1,v}\right\rangle \left\langle \widetilde{1,v}\right| + \int_0^\infty d\omega' \; e^{i\omega' t} \left|\overline{0,\omega'}\right\rangle \left\langle \widetilde{0,\omega'}\right|\right).\end{aligned} \tag{48}$$

We have only considered the terms of the identity that correspond to zero-particle and one-particle subspaces, since, from the conservation of the number of quanta, there is no contribution of other terms. We emphasize that no approximations were carried out up to here.

[9] The difference in the conventions with respect to the use of $+$ and $-$ is the following. In operators $+$ and $-$ are related with the analytic continuations for $\pm i\varepsilon$, while $+$ and $-$ in spaces are associated with the time evolution, which is only well defined for positive or negative times, respectively.

Once we have the time evolution of the density operator, the following step is to get the reduced density operator, tracing over the basis corresponding to the environment,

$$\rho_r(t) = tr_E\rho(t) = \sum_{m=0}^{\infty}\int_0^{\infty}...\int_0^{\infty} d\omega_1...d\omega_m d\omega_1'...d\omega_m'$$
$$\times \langle\omega_1...\omega_m|\,\rho(t)\,|\omega_1'...\omega_m'\rangle\,\delta(\omega_1-\omega_1')...\delta(\omega_m-\omega_m'), \tag{49}$$

where the $m=0$ term means $\langle v|\,\rho(t)\,|v\rangle$.

As we have said, the contribution of terms $m=2,3,...$ vanishes. Therefore, using the relations between "new" and "old" bases [Eqs. (25) and (26)] and the conservation of trace, we obtain a positive definite reduced density operator:

$$\rho_r(t) = c_{11}P(t)\,|1\rangle\,\langle 1| + c_{10}\Delta_0(t)\,|1\rangle\,\langle 0| +$$

$$c_{01}\Delta_0^*(t)\,|0\rangle\,\langle 1| + \{c_{00}+c_{11}\,[1-P(t)]\}\,|0\rangle\,\langle 0|\,, \tag{50}$$

where

$$\Delta_0(t) = \frac{e^{-iz_0t}}{\alpha_+'(z_0)} + \int_0^{\infty} d\omega\; e^{-i\omega t}\frac{\lambda^2 g^2(\omega)}{\eta_+(\omega)\alpha_-(\omega)}, \tag{51}$$

and $P(t) = |\Delta_0(t)|^2$ is the survival probability of the state with only one quantum in the discrete part.

We can write $P(t)$ as the sum of four terms where the first one, $\frac{e^{-\gamma t}}{|\alpha_+'(z_0)|^2}$, shows an exact exponential behavior. Expanding $|\alpha_+'(z_0)|^{-2}$ as $1+O(\lambda^2)$, we split the probability into two terms, one containing the purely exponential contribution and the other that we call "background," giving rise to derivations from that purely exponential decay law, so that

$$P(t) = e^{-\gamma t} + \text{background}. \tag{52}$$

If we take a time neither very short nor very long, the background will be smaller than the purely exponential term (for $\lambda \ll 1$) and it can be neglected, which leads to an exponential decay-law. This is not true for short times since $\frac{dP}{dt}(0) = 0$, which leads to the so-called Zeno effect [10]. For very long times the exponential term will decay faster than the background

will do, which is known as Khalfin effect [20]. We can force $P(t)$ to have an exponential appearance by defining the decay rate $\Gamma(t)$ to be time dependent, namely

$$P(t) \equiv e^{-\Gamma(t)t}, \tag{53}$$

with

$$\Gamma(t) = \gamma - \frac{1}{t}\ln\left(1 + e^{\gamma t}\text{ background}\right).$$

Obviously for an intermediate time the background can be neglected and $\Gamma(t) \simeq \gamma$. The main restriction, imposed by the Zeno period, is

$$\frac{dP}{dt}(0) = -\Gamma(0) = 0. \tag{54}$$

For a very long time the decay probability has also a non-exponential contribution as a consequence of the semi-finiteness of the energy spectrum. From papers [6,7] we know that the survival amplitude goes to zero as t goes to infinity as a consequence of the Riemann-Lebesgue theorem. Then the behavior of Δ_0 depends on the small-frequency behavior of $g^2(\omega)$. For small ω, $\alpha_+(\omega) \sim -\omega_0$, where condition (8) was also used. Then the form of Δ_0 depends essentially on $g(\omega)$ for large t. As an example we consider the case where $g^2(\omega) \sim \omega^n \exp\left(-\frac{\omega^2}{\Lambda^2}\right)$, where Λ is a cutoff [see paragraph before Eq. (19)]. By evaluating the survival amplitude we have

$$\Delta_0(t) = \lambda^2 \int_0^\infty d\omega \frac{g^2(\omega)}{|\alpha_+(\omega)|^2} e^{-i\omega t} \sim \int_0^{1/t} d\omega\ \omega^n e^{-i\omega t} \exp\left(-\frac{\omega^2}{\Lambda^2}\right),$$

where the contribution of high-frequency terms is negligible. Performing the change of variables $\omega t = x$, we obtain

$$\Delta_0(t) \sim t^{-(n+1)} \int_0^1 dx\ x^n e^{-ix}\left(1 - \frac{x^2}{\Lambda^2 t^2} + \frac{x^4}{\Lambda^4 t^4} + ...\right). \tag{55}$$

We can see that the survival amplitude merges into an algebraic long-time tail. The first relevant contribution behaves as t^{-n-1}. As a consequence the decay rate for long times must behave as

$$\Gamma(t) \propto \frac{\ln t}{t}. \tag{56}$$

The behavior at short times and intermediate times coincides with those obtained in Ref. [17]. In Fig. 2 we show the qualitative behavior of $P(t)$. Zeno's time, t_Z, and Khalfin's time, t_K, are not in scale in the picture in order to show the three different contributions to the decay probability.

Eq. (50) is the exact solution to the proposed problem, without taking any approximation. The first, second, and third terms will vanish for $t \to \infty$; in fact $P(t \to \infty) = 0$ and the same happens for $\Delta_0(t)$ (recall that $P = |\Delta_0|^2$). The first term of (51) has the factor $e^{-\frac{\gamma}{2}t}$ and the second one will tend to zero because of the Riemann-Lebesgue theorem. The probability of having the vacuum will grow in time. This means that all quanta in the discrete spectrum, except the ground state, decay into the continuum. So we find the equilibrium reduced density operator

$$\rho_* = (c_{00} + c_{11}) \left|0\right\rangle \left\langle 0\right| = tr\rho \left|0\right\rangle \left\langle 0\right| = \left|0\right\rangle \left\langle 0\right|. \tag{57}$$

As expected, the equilibrium state is the vacuum, namely for $t \to \infty$ there are no quanta in the discrete spectrum, because the initial quantum has decayed into the bath (the discrete oscillator has spread its energy over the infinite oscillators of the bath with a distribution centered at the shifted frequency ω_0) [5,15,6]. This means that the discrete harmonic oscillator has thermalized at $T = 0$. A similar result was recently obtained in Ref. [36].

In order to check the compatibility of the solution found in Eq. (50) we first briefly sketch the main points of the derivation we have done. We have obtained the exact solution of the Liouville equation. As a particular case, we have considered an initial condition restricted to the zero- and one-particle sectors and we have traced this solution over the environment modes. In that case, the survival probability $P(t)$ in Eq. (50) can be approximated by an exponential behavior, when the background contribution is neglected. Now the solution for $\rho_r(t)$ obtained through this approximation can be derived from a master equation of a Lindblad's form, where the Lindblad generator is proportional to the destruction operator a,

since in our case we are only considering a zero-temperature bath in the case of a damping motion caused by friction [37]:

$$\dot{\rho}_r(t) = -i\Omega_0 \left[a^\dagger a, \rho_r\right] + \frac{\gamma}{2}\left(2a\rho_r a^\dagger - a^\dagger a \rho_r - \rho_r a^\dagger a\right).$$

The probability of finding n quanta follows a Pauli master equation:

$$\frac{\partial}{\partial t}\langle n|\,\rho_r\,|n\rangle = \gamma\left[(n+1)\langle n+1|\,\rho_r\,|n+1\rangle - n\langle n|\,\rho_r\,|n\rangle\right]. \tag{58}$$

It is easy to see that ρ_r of Eq. (50), when the background is neglected, is solution of the Pauli equation (58).[10]

The results listed above are well known, but shown that our formalism works as the usual well established theories on the subject.

VI. SEMIGROUPS, WIGNER TIME-INVERSION AND IRREVERSIBILITY

One of the features of the system we are studying is its irreversible evolution; the system reaches the equilibrium at the far future, and of course the inverse evolution is not possible anymore. These properties were already found in Refs. [5,21] and here they are briefly reviewed.

The presence of a time-asymmetric behavior can be shown in two different ways: As the splitting of the usual evolution group in two semigroups or as the impossibility to make a time inversion. We consider that the second one is the most eloquent.

---

[10] Moreover $\rho_r(t)$ is possitive definites since this condition is equivalent to $\rho_{11}\rho_{00} \geq |\rho_{10}|^2$ ($\rho_{ij} = \langle i|\,\rho_r\,|j\rangle$) which is obviously satisfied by $\rho_r(t)$ provided it is satisfied by $\rho_r(0)$.

A. Semigroups

Physical states can be chosen to be in test space Φ_+, as we have done (or, with a simple and physically irrelevant change of convention in space Φ_-), while corresponding generalized eigenvectors are in its dual $\Phi_+^\times$ (or $\Phi_-^\times$). The proof is simple. From the Paley-Wiener theorem [9] the following lemma can be deduced:

If $f(\omega) \in H^2_+$, $e^{i\omega t} f(\omega) \in H^2_+$ only for positive times. Similarly if $f(\omega) \in H^2_-$, $e^{i\omega t} f(\omega) \in H^2_-$ only for negative times.

The asymmetry in Hardy spaces can be immediately seen from the Fourier transform representation. From the Paley-Wiener theorem it is known that Hardy class functions from above can be represented as

$$f(\omega) = \frac{1}{\sqrt{2\pi}} \int_0^\infty ds \; e^{i\omega s} \widehat{f}(s),$$

where

$$\widehat{f}(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty d\omega \; e^{-i\omega s} f(\omega).$$

The Fourier transform of Hardy class functions H^2_+ is in the space of square integrable functions with support on the positive real axis; and as the Fourier transform of $e^{i\omega t} f(\omega)$ is given by

$$\mathcal{F}\{e^{i\omega t} f(\omega)\} = \widehat{f}(s - t),$$

i.e. a function with support on $[0, \infty)$ is transformed into a function with support on $[t, \infty)$. For a negative time this last function has no longer support on $[0, \infty)$ and therefore $e^{i\omega t} f(\omega)$ does not belong to H^2_+. Analogously it can be proved the same property for the Hardy class H^2_-.

To simplify, we analyze the one-particle case; generalization to n-particle states is straightforward. Let $\phi(\omega)$ be a function in $\theta \left[\mathcal{S} \cap H^2_\pm\right]$ such that, as a consequence of previous lemma, $e^{i\omega t}\phi(\omega)$ will also be in the same space for $t > 0$ only. As $\phi(\omega) = \langle\omega|\phi\rangle$ then we can write in Dirac notation

$$e^{i\omega t}\phi(\omega) = e^{i\omega t}\langle\omega|\phi\rangle = \langle\omega|e^{-iH^{(-)}t}|\phi\rangle, \tag{59}$$

and taking into account Eq. (B5) we can state that if $|\phi\rangle \in \Phi_+$ then $e^{-iH^{(-)}t}|\phi\rangle \in \Phi_+$ only for $t > 0$. In the same way, if $|\phi\rangle \in \Phi_-$ then $e^{-iH^{(+)}t}|\phi\rangle \in \Phi_-$ for $t < 0$. Then if we postulate that Φ_+ is the space of physical states and that the physical evolution brings physical states into physical states, it turns out that this evolution only take place in the period $t > 0$, so irreversibility naturally appears [11].

B. The Wigner time-inversion

The Wigner time-inversion operator acts in a real representation as [38]

$$K\varphi(x) = \varphi^*(x). \tag{62}$$

[cf. Eq. (3)]. Then, as the complex conjugate of the functions of H^2_+ are the functions of H^2_- we have that [21]

$$\begin{aligned} K &: \Phi_+ \rightarrow \Phi_- \neq \Phi_+, \\ K &: \Phi_- \rightarrow \Phi_+ \neq \Phi_-. \end{aligned} \tag{63}$$

---

[11] Moreover, states in Φ_+ are linear combination of generalized vectors $|n, \widetilde{\omega_1 ... \omega_m}\rangle$, and these vectors evolve as

$$e^{-iH^{(-)}t}\,|n, \widetilde{\omega_1 ... \omega_m}\rangle = e^{-i(\omega_0 n + \omega_1 + ... + \omega_m)t} e^{-\frac{\gamma}{2}nt}\,|n, \widetilde{\omega_1 ... \omega_m}\rangle\,. \tag{60}$$

Therefore, with the exception of $n = 0$, they decay towards increasing t. Then we say that physical vectors in Φ_+ decay towards positive time (except the states belonging to $\Phi_+ \cap \Phi_-$, as the vacuum state $|0\rangle \bigotimes |v\rangle$, which does not decay). Analogously

$$e^{-iH^{(+)}t}\,|\overline{n, \omega_1 ... \omega_m}\rangle = e^{-i(\omega_0 n + \omega_1 + ... + \omega_m)t} e^{\frac{\gamma}{2}nt}\,|\overline{n, \omega_1 ... \omega_m}\rangle\,, \tag{61}$$

and we can say that the unphysical states in Φ_- decay towards negative times.

Then the Wigner operator is not well defined either within Φ_+ or within Φ_-, so those states in Φ_+ or Φ_- are not, in general, $t-$symmetric. Therefore if we consider that only the states of Φ_+ are "physical" or "admissible" the Wigner time inversion transforms these states in "unphysical" ones and therefore it turns out to be impossible since unphysical states simply do not exist in nature. Then, through this mathematical structure, irreversibility is incorporated in our theory.

Nevertheless $(\Phi_+ \cap \Phi_-)$ is not an empty set [14], so the time-inversion operator will be well defined there

$$K : (\Phi_+ \cap \Phi_-) \to (\Phi_+ \cap \Phi_-), \tag{64}$$

and these states will describe reversible processes, which will be $t-$symmetric.

All these things lead us to the postulate of the introduction: "Physical states are in Φ_+ (or Φ_-)." In fact, this postulate provides a mathematical structure to deal with irreversible process, as it was also shown in papers [21,5–8]. The choice between Φ_+ and Φ_- is conventional and does not lead to physical consequences, but once we choose one of these spaces the distinction between past and future becomes substantial. Moreover, if we take into account the global structure of the universe, this choice can be motivated from the asymmetry of this structure. ([12], [13])

VII. CONCLUDING REMARKS

We outline the main results of this work.

We have diagonalized the full Hamiltonian of our model and extended it in such a way that the solution is analytic in the interaction parameter. A rigorous mathematical formalism can be introduced in order to deal with unstable quantum systems (see the appendices).

Using this formalism, we have obtained a second quantized version of the decay of unstable systems and we have found the corresponding creation and annihilation operators of unstable states.

By means of a simple example, the exact time evolution of the reduced density matrix at zero temperature has been studied. We have obtained an exponential decay approach of $P(t)$ to the asymptotic value $P(t \to \infty) = 0$ which is expected when the particle is in thermal equilibrium with a zero temperature bath. For short times we have found a quadratic behavior for the decay probability $P(t)$ (Zeno effect). This short time deviation from the exponential decay law was recently measured by first time (see Ref. [39]). Other deviations from the exponential decay law, in this case for long times, naturally arise in our framework: Khalfin effect, which unfortunately are in practice far enough of any observational time scale.

VIII. ACKNOWLEDGMENTS

We acknowledge the very helpful discussions with Edgardo García Alvarez and Roberto Laura. This work was partially supported by grants CI1*-CT94-0004 and PSS* 0992 of European Union, PID 3183/93 of CONICET, EX053 of Universidad de Buenos Aires, and 12217/1 of Fundación Antorchas.

IX. APPENDICES

APPENDIX A: CREATION AND ANNIHILATION OPERATORS

Consider the annihilation and creation "unsmeared operators," b_ω and $b_\omega^\dagger$, respectively, that we have used in our calculations. Commonly they are introduced in the mathematical framework of quantum field theory by virtue of expressions like [40]

$$b\left(\phi\right) = \int \phi\left(\omega\right) b_\omega \, d\omega \tag{A1}$$

or

$$b^{\dagger}(\phi) = \int \phi^{*}(\omega)\, b_{\omega}^{\dagger}\, d\omega, \tag{A2}$$

where $b(\phi)$ and $b^{\dagger}(\phi)$ are the (well-defined or "smeared") annihilation and creation operators of the one-particle state $\phi \in \mathcal{H}$ (being $\mathcal{H}$ a Hilbert space) and their action is interpreted as the annihilation, respectively the creation, of a spectrum localized quanta, represented by a Dirac delta centered in the real value ω. In this context we find that

$$b^{\dagger}(\phi) : \bigoplus_{n=0}^{\infty} sym\left(\mathcal{H}^{\otimes n}\right) \to \bigoplus_{n=0}^{\infty} sym\left(\mathcal{H}^{\otimes n}\right),$$

$$b^{\dagger}(\phi)\,\Phi = b^{\dagger}(\phi)\left(\phi_0, \phi_1, \ldots, \phi_n, \ldots\right) = \left(0, b_0^{\dagger}(\phi)\,\phi_0, b_1^{\dagger}(\phi)\,\phi_1, \ldots, b_n^{\dagger}(\phi)\,\phi_n, \ldots\right), \tag{A3}$$

where

$$\begin{aligned} b_n^{\dagger}(\phi)\,\phi_n &= b_n^{\dagger}(\phi)\left[sym\left(\phi_n^{(1)} \otimes \phi_n^{(2)} \otimes \cdots \otimes \phi_n^{(n)}\right)\right] \\ &= sym\left(\phi_n^{(1)} \otimes \phi_n^{(2)} \otimes \cdots \otimes \phi_n^{(n)} \otimes \phi\right) \end{aligned} \tag{A4}$$

and $b(\phi) = \left[b^{\dagger}(\phi)\right]^{\dagger}$.

Of course, in the framework of the Hilbert space foundation of quantum mechanics "definitions" (A1) and (A2) are strictly formal.

It is evident that these equations are analogous to the formal definition of the Dirac delta

$$\phi(x) = \int \phi(\omega)\,\delta(x-\omega)\; d\omega, \tag{A5}$$

so, as in the case of the Dirac delta, the rigorous meaning of the "unsmeared operators" b_{ω} and $b_{\omega}^{\dagger}$ must be found in the rigged Hilbert space formulation of quantum mechanics.

One way to do this is considering the explicit definitions of b_{ω} and $b_{\omega}^{\dagger}$ as distribution valued operators. Being $\mathcal{S} \subset \mathcal{H} \subset \mathcal{S}^{\times}$ a rigged Hilbert space, we have that

$$b_{\omega} : \bigoplus_{n=0}^{\infty} sym\left(\mathcal{S}^{\otimes n}\right) \to \left[\bigoplus_{n=0}^{\infty} sym\left(\mathcal{S}^{\otimes n}\right)\right]^{\times},$$

$$b_{\omega}\Phi = b_{\omega}\left(\phi_0, \phi_1, \ldots, \phi_n, \ldots\right) = \left(b_{1\omega}\phi_1, b_{2\omega}\phi_2, \ldots, b_{n\omega}\phi_n, \ldots\right), \tag{A6}$$

where

$$b_{n\omega}\phi_n = b_{n\omega}\left[sym\left(\phi_n^{(1)}\otimes\phi_n^{(2)}\otimes\cdots\otimes\phi_n^{(n)}\right)\right] \tag{A7}$$

and $b_{n\omega}$ acts as in its definition given in Sec. II. Also

$$b_\omega^\dagger : \bigoplus_{n=0}^{\infty} sym\left(\mathcal{S}^{\otimes n}\right) \to \left[\bigoplus_{n=0}^{\infty} sym\left(\mathcal{S}^{\otimes n}\right)\right]^\times ,$$

$$b_\omega^\dagger\Phi = b_\omega^\dagger\left(\phi_0,\phi_1,\ldots,\phi_n,\ldots\right) = \left(0, b_{0\omega}^\dagger\phi_0, b_{1\omega}^\dagger\phi_1,\ldots,b_{n\omega}^\dagger\phi_n,\ldots\right), \tag{A8}$$

where

$$\begin{aligned} b_{n\omega}^\dagger\phi_n &= b_{n\omega}^\dagger\left[sym\left(\phi_n^{(1)}\otimes\phi_n^{(2)}\otimes\cdots\otimes\phi_n^{(n)}\right)\right] \\ &= sym\left(\phi_n^{(1)}\otimes\phi_n^{(2)}\otimes\cdots\otimes\phi_n^{(n)}\otimes\delta_\omega\right). \end{aligned} \tag{A9}$$

Observe that, with these definitions, only the normal product is well defined and that the "†" symbol is merely a convenient notation, that is, there is not a rigorous Hermitian conjugation involved. But if we want to define the canonical commutation relations we will be in troubles, because the product of functionals is not uniquely defined. So this is not the right way either.

Fortunately, there is another way to define the annihilation and creation "operators," b_ω and $b_\omega^\dagger$. This point of view is more abstract than the previous one. Remember that the Dirac delta can be considered as a tempered distribution, i.e. a continuous linear functional on the one-particle regular states space $\mathcal{S}$. In an analogous way the annihilation and creation "operators," b_ω and ${b_\omega}^\dagger$, and all the respective "products" can be considered as continuous linear functionals on the Canonical Commutation Relations Algebra $= CCR\left(\mathcal{S}\right)$.

We summarize some properties which characterize this kind of algebras [41]. Remember that, being $\mathcal{S}$ a nuclear metrizable space, there exists a non-decreasing basis $\{p_\alpha\}_{\alpha\in I}$ of continuous seminorms such that each seminorm is Hilbertian. Let us denote by $\mathcal{H}_\alpha$ the Hilbert space which is the completion of the quotient space $\mathcal{S}/Ker\left(p_\alpha\right)$ with respect to the quotient norm $\widehat{p}_\alpha = p_\alpha/Ker\left(p_\alpha\right)$, i.e. the space of equivalence classes defined by

$$\phi_\alpha = \left\{\chi\in\mathcal{S} : p_\alpha\left(\phi-\chi\right)=0\right\},$$

where

$$\widehat{p}_\alpha : \mathcal{S}/Ker\,(p_\alpha) \to \mathbf{R}_+,$$

$$\widehat{p}_\alpha\,(\phi_\alpha) = p_\alpha\,(\phi)\,.$$

The *-algebra $CCR\,(\mathcal{S})$ is defined as the Hausdorff projective limit [42] of the collection $\{CCR\,(\mathcal{H}_\alpha)\}_{\alpha\in I}$, where $CCR\,(\mathcal{H}_\alpha)$ is the C*-algebra [43] generated by the family of operators $\{b\,(\phi) : \phi \in \mathcal{H}_\alpha\}$, with respect to the mappings that inject each $CCR\,(\mathcal{H}_\alpha)$ into $CCR\,(\mathcal{H}_\beta)$ if $\alpha \geq \beta$, where the order in I is the induced by the ordering of the basis of seminorms. We can characterize the $CCR\,(\mathcal{S})$ as follows. Since every projective limit of a collection of C*-algebras is a b*-algebra, i.e. a complete symmetric *-algebra whose topology is defined by a basis of continuous submultiplicative seminorms, $CCR\,(\mathcal{S})$ is also a b*-algebra (see Ref. [44]). Moreover, we have that $CCR\,(\mathcal{S})$ is the strict inductive limit [42] of the collection $\left\{\bigoplus_{j=0}^{n} sym\,(\mathcal{S}^{\otimes j})\right\}_{n=0}^{\infty}$, so $CCR\,(\mathcal{S})$ is a nuclear strict inductive limit of a collection of Fréchet spaces or $\mathcal{LF}^*$-algebra [45]. With this we have that the algebra is complete, barreled, and nuclear.

Finally we have that $CCR\,(\mathcal{S}) \subset CCR\,(\mathcal{H}) \subset [CCR\,(\mathcal{S})]^{\times}$, which represents a generalized Gel'fand triplet. So, viewing the relations (A1) and (A2) as generalized expansions in the sense of the well known Maurin's theorem, one can identify b_ω and $b_\omega^\dagger$ as continuous linear functionals on the algebra $CCR\,(\mathcal{S})$, and we can say that they belong to $[CCR\,(\mathcal{S})]^{\times}$.

APPENDIX B: RIGGED EXTENSION

In this appendix we find a state space Φ with the required properties to implement our formalism of unstable states. In order to adequately define the vectors obtained in Sec. III we must restrict the Hilbert space, which is the basic mathematical structure of ordinary quantum mechanics. Recall that Dirac's bras are defined as linear functionals on

kets of $\mathcal{H}$.[12] These functionals belong to $\mathcal{H}^{\times}$, the topological dual of $\mathcal{H}$. But in this case $\mathcal{H}^{\times}$ is isomorphic to $\mathcal{H}$, then one works indistinguishably with kets and bras. However, if we restrict the topology in order to take a dense subset Φ of $\mathcal{H}$, we break the one-to-one correspondence between elements ϕ of Φ and continuous linear or antilinear functionals F over them. We will call Φ' the dual space of linear functions and $\Phi^{\times}$ the dual space of antilinear functional. We usually use the latter one. It leads to a triplet structure symbolized as $\Phi \subset \mathcal{H} \subset \Phi^{\times}$, where, to assure the convergence in the norm which defines the topology of Φ, we must require that $\langle\phi|F\rangle$ would be finite [9,14,40]. This space Φ will be the space of "regular" states Φ_{+}, as we have explained above. Changing the our convention it can be Φ_{-}

In our case, a necessary condition for $|\phi\rangle \in \Phi$ is that, the following expression, a generalization of Eq. (25), would have a rigorous meaning,

$$\begin{aligned}\langle\phi|\widetilde{n,v}\rangle &= \left\langle\phi\left|\left[\widetilde{a}^{(-)\star}\right]^{n}\right|0,v\right\rangle \\ &= \frac{1}{[\alpha'(z_0)]^{\frac{n}{2}}}\left[\langle\phi|nv\rangle + \lambda\int_0^{\infty} d\omega_1 \frac{g(\omega_1)}{[z_0-\omega_1]_+}\langle\phi|n-1,\omega_1\rangle + \dots \right. \\ &\left. +\lambda^n \int_0^{\infty}\dots\int_0^{\infty} d\omega_1\dots d\omega_n \frac{g(\omega_1)}{[z_0-\omega_1]_+}\cdots\frac{g(\omega_n)}{[z_0-\omega_n]_+}\langle\phi|0\omega_1...\omega_n\rangle\right].\end{aligned} \tag{B1}$$

The last term of the second member of Eq. (B1) must be well defined, so the function $\langle\phi|0,\omega_1...\omega_n\rangle = \phi_0^*(\omega_1...\omega_n)$ has to have an analytic continuation in each variable ω_i $(0 \leq i \leq n)$ to a region which includes the singularity z_0, so that the integral defines an analytic n-dimensional function evaluated at z_0.

The simplest choice for $\langle\phi|0,\omega_1...\omega_n\rangle$ which does not depend on the localization of z_0 is that $\langle\phi|0,\omega_1...\omega_n\rangle$ would be a Hardy function from below H_{-}^2 [14] for each variable. It is equivalent to

$$\langle\phi|0,\omega_1...\omega_n\rangle \in \theta(\mathcal{S}\cap H_{-}^2)^{\otimes n}. \tag{B2}$$

---

[12] $\mathcal{H}$ is the Hilbert space of the states we are considering. It can be the whole space of states or some subspace with precise physical properties, as the incoming or outgoing spaces.

This generalizes the criterium previously used for the one-particle sector [5,21,11]. From this criterium it can be proved that all the mathematical expressions above are well defined (see Appendix C). Then in order to $\langle i,\omega_1...\omega_m|\phi\rangle$ $(i+m=n,\ n\in\mathcal{N})$ be well defined, it must belong to the following function space

$$\langle i,\omega_1...\omega_m|\phi\rangle \in \bigoplus_{m=0}^{\infty}\theta\left[\mathcal{S}\cap H_+^2\right]^{\otimes m}, \tag{B3}$$

where $\mathcal{S}$ is the Schwartz space [14,40], and θ is the Heaviside step function, which gives the restriction to the positive real axis.

If we do the same in order to define $\langle\phi|\overline{n,v}\rangle$, we find another realization space

$$\langle i,\omega_1...\omega_m|\phi\rangle\ \in \bigoplus_{m=0}^{\infty}\theta\left[\mathcal{S}\cap H_-^2\right]^{\otimes m}, \tag{B4}$$

with $(i+m=n,\ n\in\mathcal{N})$. Therefore we define the following spaces

$$\Phi_\pm=\left\{\phi\ /\ \langle i,\omega_1...\omega_m|\phi\rangle\in\bigoplus_{m=0}^{\infty}\theta\left[\mathcal{S}\cap H_\pm^2\right]^{\otimes m}\right\}. \tag{B5}$$

The generalized eigenvectors belong to the dual spaces $\Phi_\pm^\times$, since they are antilinear functionals [9] on spaces (B3) and (B4),

$$\begin{aligned}&\left|i,\omega_1\widetilde{...}\omega_m\right\rangle\ \in\Phi_+^\times,\\ &\left|\overline{i,\omega_1...\omega_m}\right\rangle\ \in\Phi_-^\times.\end{aligned} \tag{B6}$$

These generalized eigenvectors fulfill the following relations (see [5])

$$\begin{aligned}&\left\langle i,\omega_1\widetilde{...}\omega_m|i',\omega_1'\widetilde{...}\omega_{m'}'\right\rangle\ =0,\\ &\left\langle\overline{i,\omega_1...\omega_m}|\overline{i',\omega_1'...\omega_{m'}'}\right\rangle\ =0,\end{aligned} \tag{B7}$$

$$\left\langle\overline{i,\omega_1...\omega_m}|i',\omega_1'\widetilde{...}\omega_{m'}'\right\rangle=\frac{\delta_{ii'}\delta_{mm'}}{(m!)^2}\sum_{\sigma\in G_p}\sum_{\tau\in G_p}\delta\left(\omega_{\sigma_1}'-\omega_{\tau_1}\right)...\delta\left(\omega_{\sigma_m}'-\omega_{\tau_m}\right). \tag{B8}$$

Eqs. (B7) are valid except for $i=i'=0$, and in this case we have

$$\left\langle 0,\widetilde{\omega_1 ... \omega_n} | 0,\widetilde{\omega'_1 ... \omega'_{n'}} \right\rangle = \frac{\delta_{nn'}}{(n!)^2} \sum_{\sigma,\tau \in G_p} \delta\left(\omega'_{\sigma_1} - \omega_{\tau_1}\right) ... \delta\left(\omega'_{\sigma_n} - \omega_{\tau_n}\right),$$

$$\left\langle \overline{0,\omega_1 ... \omega_n} | \overline{0,\omega'_1 ... \omega'_{n'}} \right\rangle = \frac{\delta_{nn'}}{(n!)^2} \sum_{\sigma,\tau \in G_p} \delta\left(\omega'_{\sigma_1} - \omega_{\tau_1}\right) ... \delta\left(\omega'_{\sigma_n} - \omega_{\tau_n}\right). \tag{B9}$$

G_p is the group of permutations. Eqs. (B7) say that the norm of generalized eigenvectors is zero (except $i = 0$), which is a necessary fact to conserve energy [21,5]. It is not contradictory to have null-norm vectors because these are generalized vectors which are not in the usual Hilbert space and have an underlying indefinite metric structure [21]. If we define the spaces $\mathcal{H}_+$ and $\mathcal{H}_-$ as the spaces Φ_- and Φ_+ where the condition about $\mathcal{S}$ is not required, we have

$$\mathcal{H}_\pm = \left\{ \phi \ / \ \langle n, \omega_1 ... \omega_m ... | \phi \rangle \in \bigoplus_{m=0}^{\infty} \theta \left[H^2_\pm \right]^{\otimes m} \right\},$$

then we arrive to the triplets under Eq. (33) and in Eq. (B10).[13]

Using Eq. (B5), we find a double structure of rigged Hilbert spaces for our model,

$$\Phi_+ \subset \mathcal{H}_+ \subset \Phi_+^\times,$$

$$\Phi_- \subset \mathcal{H}_- \subset \Phi_-^\times. \tag{B10}$$

APPENDIX C: THE DOUBLE INTEGRAL THEOREM

In Ref. [11] it was demonstrated that if we want that the integral

$$\int_{\mathbf{R}^+} d\omega \frac{g(\omega)\langle \phi | \omega \rangle}{z_0 - \omega} \tag{C1}$$

would be well defined, it is sufficient that

$$\langle \phi | \omega \rangle \in \theta(\mathcal{S} \cap H^2_-). \tag{C2}$$

[13]In this way $\Phi_\pm$ is dense in $\mathcal{H}_\pm$ which is the outgoing (incoming) space [46]. The $\mathcal{H}$ cited in paper [5] is actually the outgoing space $\mathcal{H}_+$.

In the two variables case it is the integral

$$\int_{\mathbf{R}^+} d\omega \frac{g(\omega)}{z_0-\omega} \int_{\mathbf{R}^+} d\omega' \frac{g(\omega')\langle\phi|\omega,\omega'\rangle}{z_0-\omega} \tag{C3}$$

the one that must be well defined. In this case we prove the following theorem.

Theorem.

If

$$\phi(\omega,\omega') = \langle\phi|\omega,\omega'\rangle \in \theta(\mathcal{S}\cap H^2_-)^{\otimes 2}, \tag{C4}$$

then integral (C3) is well defined.

Proof.

If condition (C4) is fulfilled, as $\mathcal{S}$ is a Fréchet space[14] we have ([42], page 459)

$$\phi(\omega,\omega') = \sum_{i=0}^{\infty} \lambda_i \phi_1^i(\omega)\phi_2^i(\omega'), \tag{C5}$$

where $\sum_{i=0}^{\infty} |\lambda_i| < 1, \phi_1^i(\omega),\ \phi_2^i(\omega) \in \theta(\mathcal{S}\cap H^2_-)$ $(i=1,2,...)$, $\phi_1^i,\ \phi_2^i \to 0$ when $i\to\infty$, and the r.h.s. of Eq. (C5) is absolutely convergent, namely the series

$$\sum_{i=0}^{\infty} p(\lambda_i \phi_1^i \phi_2^i)$$

is convergent for any continuous seminorm p over $\theta(\mathcal{S}\cap H^2_-)^{\otimes 2}$. Let us now define the seminorm p_{z_0} as

$$p_{z_0}(\phi) = D^2 \int_{\mathbf{R}^+} d\omega \frac{|g(\omega)|}{|z_0-\omega|} \int_{\mathbf{R}^+} d\omega' \frac{|g(\omega')||\langle\phi|\omega,\omega'\rangle|}{|z_0-\omega'|}, \tag{C6}$$

where D is the distance from $\mathbf{R}_+$ to z_0 $(D=\gamma/2)$. We must demonstrate that p_{z_0} is a continuous seminorm. We use the Hölder inequality [47]

$$\left\| \frac{g(\omega)}{z_0-\omega} \frac{g(\omega')\langle\phi|\omega,\omega'\rangle}{z_0-\omega} \right\|_1 = D^{-2} p_{z_0}(\phi) \le \|g(\omega)g(\omega')\phi(\omega,\omega')\|_1 \left\| \frac{1}{(z_0-\omega)(z_0-\omega')} \right\|_\infty \tag{C7}$$

for any $z_0 \in \mathbf{C}_-$ (the lower half-plane).[15] Since

---

[14] A Fréchet space is a metrizable complete space.

[15] We remember that

$$\left\| \frac{1}{(z_0-\omega)(z_0-\omega')} \right\|_\infty = \sup\left\{ \left| \frac{1}{(z_0-\omega)(z_0-\omega')} \right| ; \omega, \omega' \in \mathbf{R}_+ \right\} = D^{-2} \tag{C8}$$

Equation (C7) reads

$$p_{z_0}(\phi) \leq \|g(\omega)g(\omega')\phi(\omega,\omega')\|_1 \tag{C9}$$

for any $z_0 \in \mathbf{C}_-$. Then $p_{z_0}(\phi)$ is not only a continuous seminorm but also a continuous norm over $\theta(\mathcal{S} \cap H^2_-)^{\otimes 2}$. Then

$$\sum_{i=0}^{\infty} p_{z_0}[\lambda_i \phi_1^i(\omega)\phi_2^i(\omega')] = D^2 \sum_{i=0}^{\infty} |\lambda_i| \int_{\mathbf{R}^+} d\omega \frac{|g(\omega)|}{|z_0-\omega|} \int_{\mathbf{R}^+} d\omega' \frac{|g(\omega')||\phi_1^i(\omega)||\phi_2^i(\omega')|}{|z_0-\omega'|}$$

$$= D^2 \sum_{i=0}^{\infty} \int_{\mathbf{R}^+} d\omega \int_{\mathbf{R}^+} d\omega' \left| \lambda_i \frac{g(\omega)}{z_0-\omega} \frac{g(\omega')}{z_0-\omega'} \phi_1^i(\omega)\phi_2^i(\omega') \right| < \infty. \tag{C10}$$

So, calling

$$f_i^{z_0}(\omega,\omega') = \lambda_i \frac{g(\omega)}{z_0-\omega} \frac{g(\omega')}{z_0-\omega'} \phi_1^i(\omega)\phi_2^i(\omega'), \tag{C11}$$

from the corollary of the Lebesgue theorem ([48], page 33) we know that, if the series

$$\sum_{i=0}^{\infty} f_i^{z_0}(\omega,\omega') < \infty$$

converge a.e. in $\mathbf{R}_+ \times \mathbf{R}_+$, then the series, considered as a function of (ω,ω'), belongs to L_1 and

$$\int_{\mathbf{R}^+} d\omega \int_{\mathbf{R}^+} d\omega' \sum_{i=0}^{\infty} f_i^{z_0}(\omega,\omega') = \sum_{i=0}^{\infty} \int_{\mathbf{R}^+} d\omega \int_{\mathbf{R}^+} d\omega' f_i^{z_0}(\omega,\omega'). \tag{C12}$$

Then, going back to Eq. (C3) we have

$$\|f\|_1 = \int_{\mathbf{R}_+} |f(\omega)| d\omega,$$

$$\|f\|_\infty = \sup\{|f(\omega)|; \omega \in \mathbf{R}_+\},$$

and also that $\mathcal{S} \in L_1$, i.e. any Schwartz function is integrable.

$$\int_{\mathbf{R}^+} d\omega \frac{g(\omega)}{z_0-\omega} \int_{\mathbf{R}^+} d\omega' \frac{g(\omega')\langle\phi|\omega,\omega'\rangle}{z_0-\omega} = \int_{\mathbf{R}^+} d\omega \frac{g(\omega)}{z_0-\omega} \int_{\mathbf{R}^+} \sum_{i=0}^{\infty} \lambda_i \phi_1^i(\omega)\phi_2^i(\omega')$$

$$= \sum_{i=0}^{\infty} \lambda_i \int_{\mathbf{R}^+} d\omega \frac{g(\omega)\phi_1^i(\omega)}{z_0-\omega} \int_{\mathbf{R}^+} d\omega' \frac{g(\omega')\phi_2^i(\omega')}{z_0-\omega}. \tag{C13}$$

The l.h.s. of Eq. (C13) is well defined since it is an integral of a L_1 function. Moreover it is the sum of products of two well defined integrals, like the one of Eq. (C1), since from hypothesis $\phi_1^i(\omega), \phi_2^i(\omega') \in \theta(\mathcal{S} \cap H_-^2)$. Thus, the proof is complete.

Of course, this theorem can be generalized from the case of two factors, to the case of n factors and, taking into account Eq. (27), it proves that Eq. (B1) is well defined if condition (B2) is fulfilled.

APPENDIX D: FIGURE CAPTIONS

Fig. 1: Deformation of the contour of integration taking into account the presence of the complex pole z_0.

Fig. 2: Behavior of the decay probability showing the Zeno, exponential, and Khalfin phases.

REFERENCES

[1] F.H. Gaioli, E.T. Garcia Alvarez, and J. Guevara, Int. J. Theor. Phys. **36**, 2167 (1997).

[2] F.H. Gaioli, E.T. Garcia Alvarez, and D.G. Arbó, Int. J. Theor. Phys. **38**, 183 (1999).

[3] E.T. Garcia Alvarez and F.H. Gaioli, Physica A **257**, 298 (1998).

[4] F.H. Gaioli and E.T. Garcia Alvarez, Physica A **264**, 338 (1999).

[5] M.A. Castagnino and R. Laura, Phys. Rev. A **56,** 108 (1997).

[6] R. Laura and M.A. Castagnino, *Functional approach for quantum systems with continuous spectrum,* Phys. Rev. E (in press, 1998).

[7] R. Laura and M.A. Castagnino, *Minimal irreversible quantum mechanics. Mixed states and diagonal singularity,* Phys. Rev. A (in press, 1998).

[8] M. Castagnino, Phys. Rev. D **57**, 750 (1998).

[9] A. Bohm, *Quantum Mechanics: Foundations and Applications* (Springer-Verlag, Berlin, 1986).

[10] E.C.G. Sudarshan, C.B. Chiu, and V. Gorini, Phys. Rev. D **18**, 2914 (1978).

[11] I. Antoniou and I. Prigogine, Physica A **192,** 443 (1993).

[12] M. Castagnino, *The global nature of the arrow of time and Bohm-Reichenbach diagram,* Proceeding G21, Goslar, 1996, H. Doebner et al. eds. (Springer-Verlag, Berlin, 1998).

[13] M. Castagnino, S. Iguri, A. Ordoñez, E. Gunzig, *Kolmogorov-Lax-Phillips systems as branch systems of a Reichenbach diagram,* Latinamerican Workshop on Dynamical Systems, E. Tirapegui ed., Valparaíso, Chile, 1997.

[14] A. Bohm and M. Gadella, *Dirac Kets, Gamow Vectors and Gel'fand Triplets* (Springer-Verlag, Berlin, 1989).

[15] F.H. Gaioli, E.T. Garcia Alvarez, and J. Guevara, Int. J. Theor. Phys. **36,** 2167 (1997);

F.H. Gaioli, E.T. Garcia Alvarez, and D.G. Arbó, *Quantum Brownian Motion. II,* Int. J. Theor. Phys. (in press, 1998).

[16] I. Joichi, S. Matsumoto, and M. Yoshimura, Phys. Rev. A **57**, 798 (1998); Prog. Theor. Phys. **98**, 9 (1997).

U. Weiss, *Quantum Dissipative Systems* (World Scientific, Singapore, 1993).

[17] B.L. Hu, J.P. Paz, and Y. Zhang, Phys. Rev. D **45**, 2843 (1992).

[18] A. Arai, J. Math. Phys. **22**, 2539 (1981); 2549 (1981).

[19] B. Misra and E.C.G. Sudarshan, J. Math. Phys. **18**, 756 (1977).

[20] L. Khalfin, Soviet Phys. JETP **6**, 1053 (1958).

[21] M.A. Castagnino, F. H. Gaioli, and E. Gunzig, Fund. Cosmic Phys. **16**, 221 (1996), and references therein.

[22] M. Castagnino and F. Lombardo, Gen. Rel. Grav. **28**, 263 (1996).

[23] R. Aquilano and M. Castagnino, Astr. Space Sci. **238**, 159 (1996).

[24] P. Manzur and E. Montroll, J. Math. Phys. **1**, 70 (1960).

[25] R.J. Rubin, J. Am. Chem. Soc. **90**, 3061 (1968).

[26] J.L. Gruver, J. Aliaga, H.A. Cerdeira, and A.N. Proto, Phys. Rev. E **51**, 6263 (1995).

[27] P. Ullersma, Physica **32**, 27 (1966).

[28] K.O. Friedrichs, Commun. Pure Appl. Math. **1**, 361 (1948). See also P. Exner, *Open Quantum Systems and Feynman Integrals* (Reidel, Amsterdam, 1985).

[29] I. Prigogine and T. Petrosky, Physica **147**A, 33 (1987).

[30] M.A. Castagnino, G. Domenech, M.L. Levinas, and N. Umérez, J. Math. Phys. **37**, 2107 (1996).

[31] C. Cohen-Tannoudji, J. Dupont-Roc, and G. Grynberg, *Atom-Photon Interactions* (Wiley, New York, 1992).

[32] I. Gel'fand and G. Shilov, *Generalized Functions,* Vol. 3 (Academic Press, New York, 1967).

[33] M.A. Castagnino, M. Gadella, F.H. Gaioli, E.T. Garcia Alvarez, and R. Laura, *The Friedrichs model. A solvable model of unstable systems (*in preparation).

[34] M.A. Castagnino, M. Gadella, F.H. Gaioli, and R. Laura, *Gamow vectors and time asymmetry (*submitted to Int. J. Theor. Phys., 1998).

[35] M.A. Castagnino and R. Laura (submitted to Phys. Rev. D, 1998).

[36] G.V. Efimov and W. von Waldenfels, Ann. Phys. (N.Y.) **233**, 182 (1994).

[37] D. Giulini *et al.*, *Decoherence and the Appearence of a Classical World in Quantum Theory* (Springer-Verlag, Berlin, 1996).

[38] A. Messiah, *Mécanique Quantique* (Dunod, Paris, 1964).

[39] S.R. Wilkinson *et al.*, Nature **387**, 575 (1997).

[40] N.N. Bogolubov, A.A. Logunov, and I.T. Todorov, *Introduction to Axiomatic Quantum Field Theory* (Benjamin, New York, 1975).

[41] S. Iguri, *La formulación de la Mecánica Cuántica en álgebras nucleares* (master thesis UBA, unpublished 1997), in Spanish.

[42] F. Treves, *Topological Vector Spaces, Distributions and Kernels* (Academic Press, New York, 1967).

[43] P. de la Harpe and F. Jones, *Introduction to C^*-algebras* (unpublished, 1996).

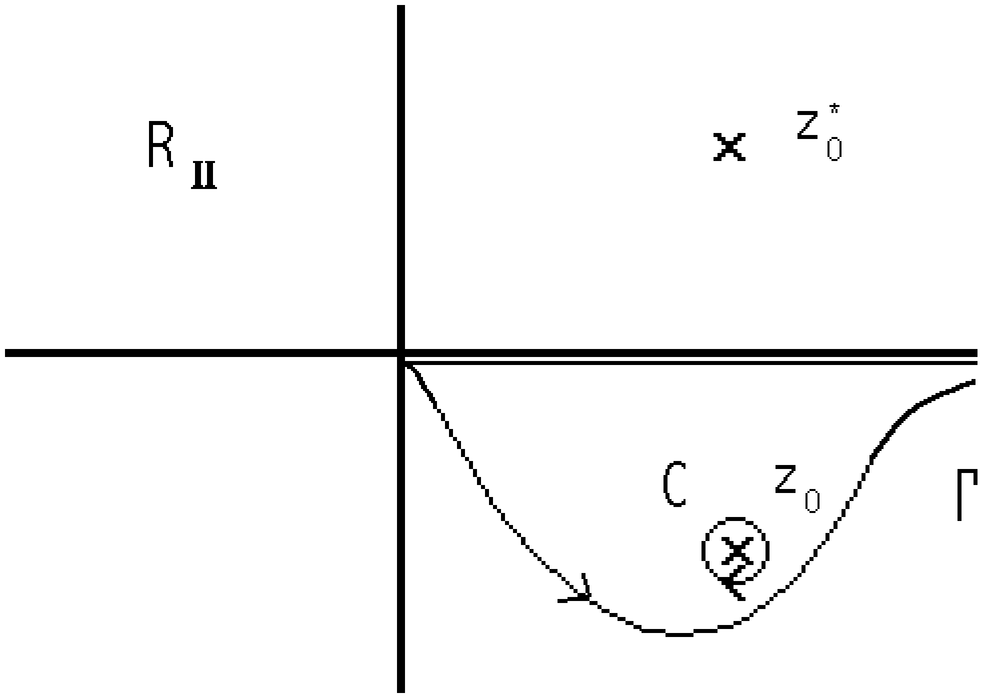
[44] S. Iguri and M. Castagnino, Int. J. Theor. Phys. **38**, 165 (1999).

[45] A. Belanger and G.F. Thomas, Can. J. Math. **42**, 410 (1990).

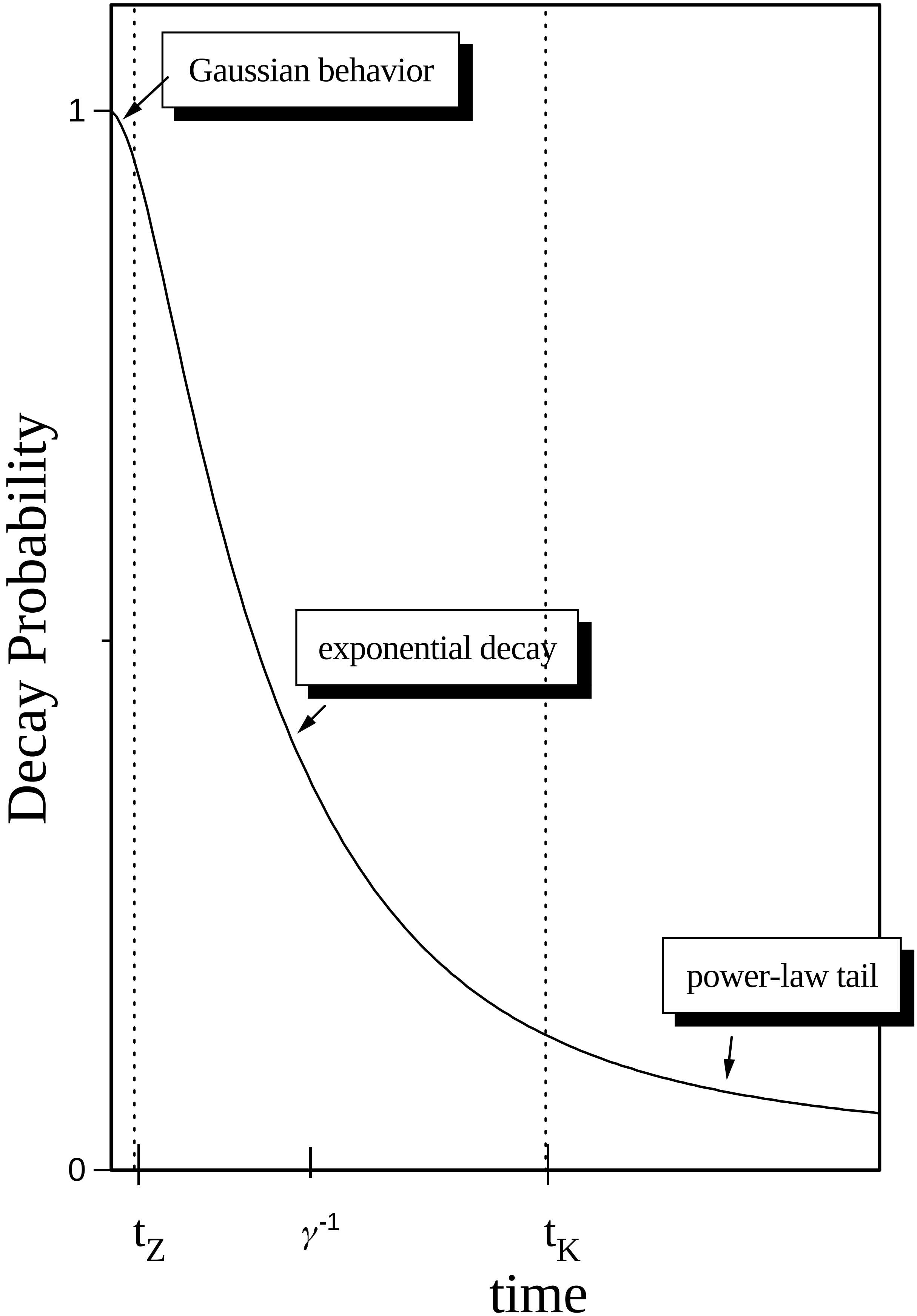
[46] P.D. Lax and R.S. Phillips, *Scattering theory* (Academic Press, New York, 1967).

[47] M. Cotlar and R. Cignoli, *Nociones de Espacios Normados* (EUDEBA, Buenos Aires, 1967), in Spanish.

[48] W. Rudin, *Functional Analysis (*MacGraw-Hill, New York, 1973).



R_{II}
z_0^*
C
z_0
Γ


Gaussian behavior
exponential decay
power-law tail
Decay Probability
1
0
t_Z
γ^-1
t_K
time